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Application of the Inflection Point in the Evaluation of the Halley and Newton-Raphson Techniques for Finding the Root of Non-Linear Equations

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Abstract. Numeric Method is one of the methods used to solve nonlinear equation roots. Many methods can be used, both open methods and closed methods. In this case, the method used is closed, namely the Newton-Raphson Method and Halley Method. The research aims to find out the comparison result between the Newton-Raphson Method and the Halley Method. The research used a literature method from a book, journal, and any other literature, where it connected with the topic. The steps used are formulation problem, finding and collecting information, describing and explaining the information, analysis, and conclusion of the result. The conclusion can be explained with a table of data and explanations Based on data analysis, it can be stated that the Halley Method is faster toward convergence compared to the Newton-Raphson Method based on the first case or second case.

Keywords: Newton-Raphson Method, Halley Method, Non-Linear Equation, convergence

1 Introduction

Numerical methods are one way to solve non-linear equations. A common problem is the solution of the root of a non-linear equation. The approach to solving mathematical problems using a set of simple arithmetic operations and logical operations on a given set of numbers or numerical data [1].

A numerical method is a method for determining numerical solutions, in which case repeated counting operations are performed to complete the numerical solution. Numerical completions are determined by performing a certain repetition procedure so that each result will be more accurate than the previous calculation. By performing repetition procedures that are considered sufficient, the estimated result is eventually obtained that is close to the exact value. The exact value can only be known if a function $f(x)$ can be solved analytically [2].

Completing the search for nonlinear equations with numerical steps can be divided into two forms of the method, namely: 1) Closed method, said closed method because the process of searching for the root of the equation is in a certain space. Examples of this closed approach are the method of table, bisexual method, and false rule method. 2) Open method, called open method because of the unlimited space of a particular space in the process of searching the roots of nonlinear equations. Examples of open methods are fixed point iteration methods, Newton-Raphson methods, and Secant methods as well as Halley methods [3].

The search for the root of the equation begins with the estimation of the first equation root, followed by the next estimation, and so on until the last estimation, which is then expressed as the root equation of that numerical calculation. The process must be convergent, i.e. the difference between the estimates before and after becomes smaller. Once considered sufficient, the process of finding the root of the equation stopped. The convergence speed of a process, that is, the speed of the process to reach the final result [4].

The point of the problem in this review is how the comparison between the Newton-Raphson Method and the Halley Method results in the determination of the root of non-linear equations. The purpose of this journal is to 1) know the form of analysis of the Newton-Raphson and Halley methods. 2) Know the solution of the roots of the non-linear equations using the Newton-Raphson and Halley methods. 3) Know the results of the comparison of the convergence between the Halley and the Newton-Raphson method.

2 Root Research Methods

2.1 Newton-Raphson Method

The Newton-Raphson method is used to solve nonlinear equations with a single variable, only if on the first derivative of $f(x)$ there is a continue on the entire solution. Method NR has the following characteristics: 1) Requires an initial intersection, 2) requires the calculation of the derivative of the function $f(x)$ in each iteration.

The characteristic of both Newton's methods relates to the fact that the next line is obtained by drawing the curve y=f(x) at the point where the previous line has an abyss to cut the x-axis. The point of cutting the line with the x-axis is the next one. The process continues until the obtained boundary qualifies for the specified accuracy.

The planning of the NR method is the approach of the curve $f'(x)$ based on the vertical line on the curve, so it requires a gradient of the line of the f^' (x) curve that can be determined by

$$f(x) = \frac{\Delta y}{\Delta x} = \frac{f(x_n) - 0}{x_n - x_{n+1}} \tag{1}$$

2.2 Halley Method

The Halley Method is one of the root-seeking algorithms for solving nonlinear equations, named after astronaut Edmund Halley. (1656-1742). Unlike the Newton-Raphson method, which usually has a quadratic convergence rate, in some cases, Halley's method can reach a cubic convergence rate[5].

The Halley method iteration formula is obtained from the decrease of the Taylor order-2 line, with little algebraic manipulation and substitution of the Newton-Raphson method so that the form of the equation is processed as follows:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) - \frac{1}{2}f''(x_n)\frac{f(x_n)}{f'(x_n)}} \tag{2}$$

2.3 Convergence

The search for the root of the equation begins with the estimation of the first equation root, followed by the next estimation, and so on until the last estimation, which is then expressed as the root equation of that numerical calculation. The process must be convergent, i.e. the difference between the estimates before and after becomes smaller. Once considered sufficient, the process of finding the root of the equation stops.

$$|x_2 - x_1| > |x_3 - x_2| > |x_4 - x_3| \ldots |x_n - x_{n+1}| \tag{3}$$

3 Research Methods

The purpose of the study is to gather information and data from sources such as books, journals, and other sources of reading. The sources used are Purcell's calculus, Rinaldi Munir's numerical method, Suprapto's linear programming, and many other sources as well as records during the course of the lecture.

This series of research processes is: 1) Determining the problem formula. 2) Collecting information or data and materials on how to explore and understand literature related to non-linear equations, Newton-Raphson methods, Halley methods, and turning points. 3) After the information on the Newton-Raphson method and the Halley method are collected, deducted, and presented in the form of an analysis of Newton-Raphson and

Halley's methods. 4) The next step, providing examples of the design of non-linear equations using Newton Raphson, and Halley methods with solutions such as the roots of the non-linear equation and the number of iterations that occurred. 5) Then, compare the convergence rates of the Newton-Raphson method and Halley method based on data obtained from the example of the solution of non-linear equations.

4 Results and Discussion

Solving Non-Linear Equation Roots

Simulation of root solving using the Newton-Raphson and Halley methods using two case examples: 1) $f(x) = x^3 - 3x^2 - 15x + 5$. and 2) $f\ (x) = 2x^3 - 6x^2 + 10 - 10$ with the following result:

Table 1. Convergence comparison Newron-Raphson Method and Halley Methods case 1

Convergence comparison						
	Newton-Raphson Method			Halley Method		
r	x_r	$f(x_r)$	ε	x_r	$f(x_r)$	ε
0	1	-12	1	1	-12	1
1	0.3333333333	-0.296296296	0.666666667	0.3333333333	-0.296296296	0.6666666667
2	0.3155555556	-0.000637717	0.017777778	0.3155175485	-0.000007008	0.0178157848
3	0.3155171263	-0.000000001	0.000028429	0.3155171262	0	0.0000004223
4	0.3155171262	0	0.000000001			

Based on the result of $f(x) = x^3 - 3x^2 - 15 + 5$ with $x_0 = 1$ dan $\varepsilon = 0{,}00001$, used Newton-Raphson method obtained root value $x = 0{,}315517162$ with the number of iterations of 4 times with details, 1) the initial estimate $x_0 = 1$ is obtained $x_1 = 0{,}3333333333$ with the value of the function $f(x_1) = -0{,}296296266$, whereas for the error is 0,6666666667, 2) the repetition is performed with a value of $x_1 = 0{,}333333333$ obtain the value $x_2 = 0{,}315555556$ and the function value of $f\ (x_2) = -0{,}000637717$ with the error of 0,017777778, 3) the repeat is done again with the amount $x_2 = 0{,}3155556$ obtained the estimated value
$x_3 = 0{,}3155171263$, for the functional value $f(\ x_3) = 0{,}000000001$ errors of 0.000028429, 4) the errors are still greater than ε, then the repetitions are performed at the value $x_3 = 0155151263$ obtain root values $x_4 = 0{,}311551262$ and the value $f(x_4) = 0$ mistakes of 0.000000001 < ε. then iteration is stopped on iterations 4th

Using the Halley method obtained the root value of $x = 0{,}315517162$ with the number of iterations of 3 times with details, 1) the initial estimate of $x_0 = 1$ obtains $x_1 = 0{,}33333333$ with the value of the function $f(x_1) = -0{,}29629626$, whereas for the error is 0.666666666667, 2) performs the repetition with the amount of $x_1 = 0.333333333$ obtaining the value $x_2 = 0.315617548$ and the function value of $f\ (x_2) = 0{,}000007008$ with the error of 0.0178157848, 3) repeats with a value of $x_2 = 0.3155755485$ obtain the estimated value of $x_3 = 0.31557171262$, for the function values of $f(x_3) = 0.0000004223$, because $|x_4 - x_3| = 0.3155175485 - 0.3155171263\ | = 0.0000004223 < \varepsilon$, then iteration is stopped at 3rd.

Based on the results of solving the roots of the equation $f(x) = 2x^3 - 6x^2 + 10 - 10$ with initial estimates $x_0 = 1$ and $\varepsilon = 0.00001$, using the Newton-Raphson method the root value $x = 0.315517162$ is obtained with the sum Iterate 5 times with details: 1) With the initial estimate $x_0 = 1$, we get $x_1 = 2$ with a function value of $f(x_1) = 2$, while the error is 1. 2) Next, we repeat with the value $x_1 = 2$, we get the value $x_2 = 1\ .8$ and the function value $f(x_2) = 0.224$ with an error of 0.2. 3) Next, repeat with the value $x_2 = 1.8$, the estimated value $x_3 = 1.771428571$ is obtained, for the function value $f(x_3) = 0.00387172$ with an error of 0.028571429. 4) Next, repetition is carried out with the value

$x_3 = 1.771428571$, the value $x_4 = 1.770917157$ and the function value $f(x_4) = 0.00000121$ with an error of 0.000511414. 5) Because the error is still greater than ε, a repetition is carried out with the value $x_4 = 1.770917157$ to obtain the root value $x_5 = 1.770916997$ and the value $f(x_5) = 0$ with an error of 0.00000016. Because $|x_5 - x_4| = |1.770916997 - 1.770917157| = 0.00000016 < \varepsilon$ then the iteration stops at the 5th iteration.

. **Table 2.** Convergence comparison Newron-Raphson Method and Halley Methods case 2

Convergence comparison						
	Newton-Raphson Method			Halley Method		
r	x_r	$f(x_r)$	ε	x_r	$f(x_r)$	ε
0	1	-4	1	1	-4	1
1	2	2	1	2	2	1
2	1.8	0.224	0.2	1.772727273	0.01371150	0.227272727
3	1.7714228571	0.00387172	0.028571429	1.770916997	0	0.001810276
4	1.770917157	0.00000121	0.000511414	1.770916997	0	0
5	1.770916997	0	0.00000016			

Using the Halley method, the root value $x = 1.770916997$ is obtained with 4 iterations. For more details, see the following details: 1) With an initial estimate of $x_0 = 1, x_1 = 2$ is obtained with a function value of $f(x_1) = 2$, whereas for error is 1. 2) Next, repeating with the value $x_1 = 2$, we get the value $x_2 = 1.772727273$ and the function value $f(x_2) = 0.01371150$ with an error of 0.227272727. 3) Next, repeating with the value $x_2 = 1.772727273$, the estimated value $x_3 = 1.770916997$ is obtained, for the function value $f(x_3) = 0$ with an error of 0.001810276. 4) Error conditions have not been met. So another iteration is carried out with the value $x_3 = 1.770916997$, we get the value $x_4 = 1.770916997$, for the function value we get $f(x_4) = 0$, the error value is 0. Because $|x_4 - x_3| = |1.770916997 - 1.770916997| = 0 < \varepsilon$ then the iteration stops at the 4th iteration.

5 Conclusion

From the results of calculations using the Newton-Raphson method in Equation $f(x) = x^3 - 3x^2 - 15x + 5$ obtained the root value of the equation $x = 0.315517126$, the iteration stops at $r = 3$ or the 4th iteration. Meanwhile, using the Halley method, the equation $f(x) = x^3 - 3x^2 - 15x + 5$ has a root value of $x = 0.315517126$ and ends at $r = 2$ or the 3rd iteration. Likewise with the results of the equation $f(x) = 2x^3 - 6x^2 + 10x - 10$, using the Newton-Raphson method, the root of the equation is obtained from the calculation results, namely $x = 1.770916997$ and $r = 4$ or the iteration stops at the 5th iteration. Meanwhile, using the Halley method, the root of the equation $x = 1.770916997$ and $r = 3$ is obtained, which means that the iteration stops at the 4th iteration. These two results were obtained using a tolerance limit of 0.00001 and the initial guess value $x_0 = 1$, the result of calculating the inflection point value at $f''(x) = 0$. With these results, it can be concluded that finding the roots of non-linear equations using the Halley method converges faster than the Newton-Raphson method. In the first case, the Halley method only requires 3 iterations while the Newton-Raphson method requires 4 iterations to reach the root value. In the second case, the Halley method only requires 4 iterations to reach the root value while the Newton-Raphson method requires 5 iterations.

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