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Modeling Network Problem using Metric Dimension: Applied Algorithm on Corona Graph

Deddy Rahmadi[1,*], Ilma Nindita Ramadhani[2], Clarissa Elva Dheana[3], Miftah Aulia Mustamin[1]

[1] Department of Mathematics, Faculty of Science and Technology, UIN Sunan Kalijaga, Yogyakarta, Indonesia
[2] Department of Physics, Faculty of Science and Technology, UIN Sunan Kalijaga, Yogyakarta, Indonesia
[3] Department of Informatics, Faculty of Science and Technology, UIN Sunan Kalijaga, Yogyakarta, Indonesia
[*]deddy.rahmadi@uin-suka.ac.id

Abstract. Let G be a graph is a finite set of vertices and edges. A graph G can be defined as a pair of sets $(V(G), E(G))$. The minimum cardinality of all distinguishing sets in a graph is called the metric dimension. The metric dimension was first introduced in 1966 by Harary and Melter. The method used in this research is deductive proof. The results obtained from this research are we determine the metric dimension of the graph resulting from the corona operation on $C_n \odot C_{n-1}$ and obtain the result that is 2n.

Keywords: Graph Theory, Modelling, Network, Metric Dimension, Corona Graph.

1 Introduction

In this paper, we consider finite, simple, and connected graphs [1]. The notation and terminologies mostly follow that of Chartrand and Oellermann [2] and Gallian [3]. Let G be a graph with a set of vertices $V(G)$ and a set of edges $E(G)$. The distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of the shortest $u - v$ path in G. For an ordered set $W = \{w_1, w_2, \cdots, w_k\} \subseteq V(G)$ of vertices, we refer to the ordered k-tuples $r(v|W) = \left(d(v, w_1), d(v, w_2), \cdots, d(v, w_k)\right)$ as the $(metric)$ representation of v concerning W. The set W is called a resolving *set* for G if $r\ (v|W\ ) = r\ (v|W\ )$ implies that $u = v$, for all $u, v \in V(G)$. A resolving set of minimum cardinality for a graph G is called a minimum resolving set or a basis for G.

The metric dimension of G, denoted by $dim(G)$, is the number of vertices in a basis for G. The papers discussing the notion of a (minimum) resolving set were initially written by Slater in [4] and [5]. Slater introduced the concept of a resolving set for a connected graph G under the term location set. He called the cardinality of a minimum resolving set by the location number of G. Independently, Harary and Melter [6] introduced the same concept but used the term metric dimension instead.

Some authors have investigated the problem of finding the metric dimension. Chartrand et. al. [7] determined the bounds of the metric dimensions for any connected graphs and determined the metric dimensions of some well-known families of graphs such as trees, paths, and complete graphs. Buczkowski et. al. [8] proved the existence of a graph G with $dim(G) = k$, for every integer $k \geq 2$. In addition, they also determined the dimensions of the wheels. Furthermore, the metric dimension problem has been investigated for regular bipartite graphs in [9], trees and grid graphs in [10], Petersen graph in [11], generalized Petersen graph in [12], join of two graphs in [13], Grassman graph in [14], and other graphs [15,16,17]. Some authors also investigated the graphs on some variants of metric dimension. Local metric dimension on line graph from friendship and strong product graph [18, 19], Strong metric dimension on some related wheel graph [20], K-metric dimension on double fan graph [21], mixed metric dimension on double fan graph [22], and double metric dimensions of cactus graphs and block graphs [23].

The study of the metric dimension on the corona product of cycle graphs has significant practical implications across various fields. One of the important applications of this concept is in communication networks and circuit design, where optimal location determination and cost minimization are crucial. A deep understanding of the metric structure of these graphs can aid in developing more efficient algorithms for routing and node placement in networks. Additionally, other applications include logistics optimization and resource management in distributed systems, where cycle graphs and their corona products can accurately represent complex relationships among entities. Thus, this research not only provides valuable theoretical contributions but also offers practical solutions to real-world problems faced in current technology and industry.

2 Research Methods

The method used in this research is deductive proof, which involves conducting case studies and then constructing a conjecture (assumption). Based on the constructed assumption, a deductive proof process is then carried out. If proven, modifications will be made to the case studies to obtain other assumptions. If the process is successful, the assumption will become a theorem, which is the result of the research.

3 Results and Discussions

A graph is a set of objects called vertices that are connected by links called edges. The following definition of a graph is given in more detail.

Definition 3.1. Given a graph G = (V, E) where vertices V(G) is a finite, nonempty set, and edges E(G) is a (non-empty) set, such that:

1. Every edge E(G) connects exactly two distinct non-consecutive vertices in V(G).
2. Every two vertices in V(G) are either connected by at least 1 edge or not connected at all.

Example 3.1. Let G = (V, E) be a graph where V = { v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8) and E ={ v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_1v_6, v_1v_7, v_3v_8, v_4v_8, v_5v_8, v_6v_7}. Therefore, with these conditions, the graph G is shown in Figure 1.

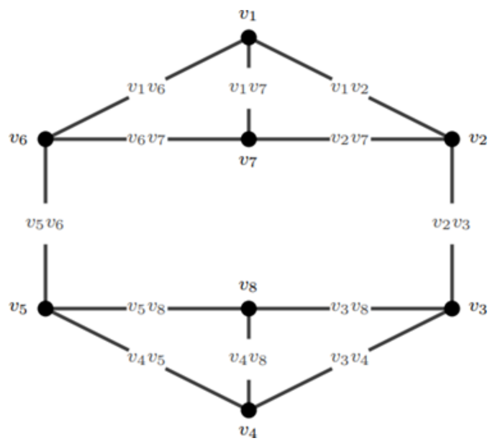

Figure 1. Graf $G = (V, E)$

After knowing the definition and examples of graphs, we will discuss cycle graphs next. The following is an explanation of cycle graphs.

Definition 3.2. Let $G = (V, E)$ be a graph with $v_1, v_2 \in V(G)$ with $|V(G)| \geq 3$ and edge $v_1v_2 \in E(G)$ is said to be cycle graph order n that denoted C_n if

$$V(G) = \{v_1, v_2, \dots, v_n\}$$
$$E(G) = \{v_1v_2, v_2v_3, v_3v_4, \dots v_nv_1\}$$

Example 3.2. Given the example of cycle graph in Figure 2.

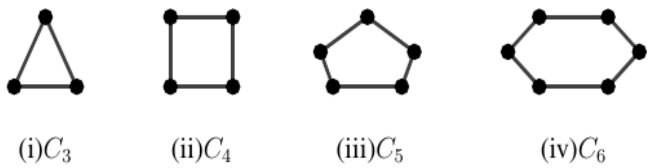

Figure 2. Cycle Graph

Metric Dimension

This part will give the terminology of metric dimension. Computing the metric dimension of graphs using the metric dimension problem (MDP) is a difficult combinatorial optimization problem. The metric dimension of a connected graph G is the minimum number of vertices in a subset B of G such that all other vertices are uniquely determined by their distances to the vertices in B. In this case, B is called a metric basis for G. The basic distance of a metric two-dimensional graph G is the distance between the elements of B. Giving a characterization for those graphs whose metric dimensions are two, they enumerated the number of n vertex metric two-dimensional graphs with the basic distance.

Definition 3.3. Let G be a connected graph and d(u,v) be the shortest path. An ordered set vertex $W = \{w1, w2, \ldots, wk\} \subset V(G)$ and representation of v over W defined as distance every vertex v to W, denoted by

$$r(v|W) = (d(v, w1), d(v, w2), \ldots, d(v, wk))$$

Set W is a resolving set for G if $r(u|W) \neq r(v|W)$ then $u \neq v$ for every two vertices u and v in G. Minimum cardinality of resolving set called metric dimension of G, denoted by dim(G).

Example 3.3. Given cycle graph order 8 with vertex set $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$ and resolving set $W = \{v_2, v_3\}$

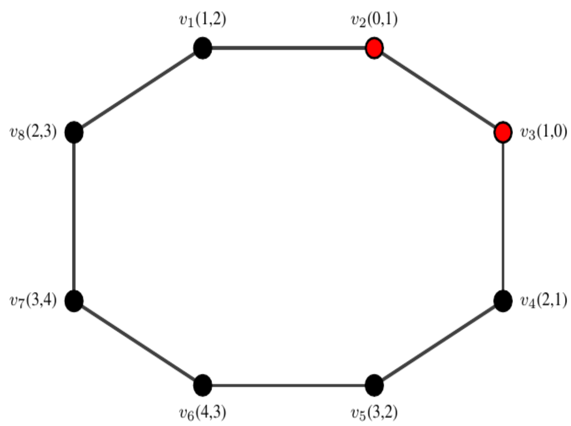

Figure 3. Cycle graph C_8

Theorem 3.1. If C_n is a cycle graph with n vertices and $n \geq 3$, then $\dim(C_n) = 2$.

Proof. Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ is the vertex set of the cycle graph with n vertices and $n \geq 3$.

For n odd. Let $W = \{v_{n-1}, v_n\}$ we have representation for every vertex of G over W i

$$r(v_1|W) = (2, 1)$$

$$r(v_2|W) = (3, 2)$$

$$r(v_3|W) = (4, 3)$$

$$\vdots$$

$$r(v_{\frac{n-3}{2}}|W) = (\frac{n-1}{2}, \frac{n-3}{2})$$
$$r(v_{\frac{n-1}{2}}|W) = (\frac{n-1}{2}, \frac{n-1}{2})$$

$$r(v_{\frac{n+1}{2}}|W) = (\frac{n-3}{2}, \frac{n-1}{2})$$
$$r(v_{\frac{n+3}{2}}|W) = (\frac{n-5}{2}, \frac{n-3}{2})$$
$$\vdots$$
$$r(v_{n-2}|W) = (1, 2)$$
$$r(v_{n-1}|W) = (0, 1)$$
$$r(v_n|W) = (1, 0)$$

Because $\forall u, v \in V(C_n), u \neq v, r(u|W) \neq r(v|W)$, then $W = \{v_n - 1, v_n\}$ is a revolving set. Then we will prove $W = \{v_n - 1, v_n\}$ is the minimum resolving set. Because C_n is cycle graph, we have $\dim(C_n) \neq 1$. Therefore, if there is no resolving set with cardinality less than 2, then W with cardinality 2 is the resolving set with minimum cardinality. We get dim $(C_n) = 2$ for n odd.

For n is even.

Let $W = \{v_{n-2}, v_n\}$ we have representation for every vertex of G over W is

$$r(v_1|W) = (2, 1)$$
$$r(v_2|W) = (3, 2)$$
$$r(v_3|W) = (4, 3)$$
$$\vdots$$
$$r(v_{\frac{n-2}{2}}|W) = (\frac{n}{2}, \frac{n-2}{2})$$
$$r(v_{\frac{n}{2}}|W) = (\frac{n-2}{2}, \frac{n}{2})$$
$$r(v_{\frac{n+2}{2}}|W) = (\frac{n-4}{2}, \frac{n-2}{2})$$
$$\vdots$$
$$r(v_n - 2|W) = (1, 2)$$
$$r(v_n - 1|W) = (0, 1)$$
$$r(v_n|W) = (1, 0)$$

Because $\forall u, v \in V(G), u \neq v, r(u|W) \neq r(v|W)$, then $W = \{v_{n-2}, v_n\}$ is resolving set.
Then we will prove $W = \{v_{n-2}, v_n\}$ is the minimum resolving set. Because C_n is cycle graph, we have $\dim(C_n) \neq 1$. Therefore, if there is no resolving set with cardinality less than 2, then W with cardinality 2 is the resolving set with minimum cardinality. We get dim $(C_n) = 2$ for n even.

Corona Operation Results Graph

Definition 3.4. The corona operation on a graph is a graph-theoretical construction where for a given graph G with vertex set $V(G)$ and edge set $E(G)$, and a second graph H with vertex set $V(H)$ and edge set $E(H)$, the corona $G \odot H$ is formed by taking one copy of G and $|V(G)|$ copies of H and then joining each vertex v in G to every vertex in the v-th copy of H. In simpler terms, it involves attaching a copy of graph H to each vertex of graph G, connecting each vertex of G to every vertex in the corresponding copy of H.

Metric Dimension on Corona Operation $C_n \odot C_{n-1}$

Example 3.4. Given the graph $C_4 \odot C_3$, which is the corona product of the cycle graph C_4 denoted by $V = \{v_1, v_2, v_3, v_4\}$ and the cycle graph C_3 denoted by $V\{u_{11}, u_{12}, u_{13}, u_{21}, u_{22}, u_{23}, u_{31}, u_{32}, u_{33}, u_{41}, u_{42}, u_{43}\}$, where $n \geq 4$. The graph $C_4 \odot C_3$ is shown in Figure 5.

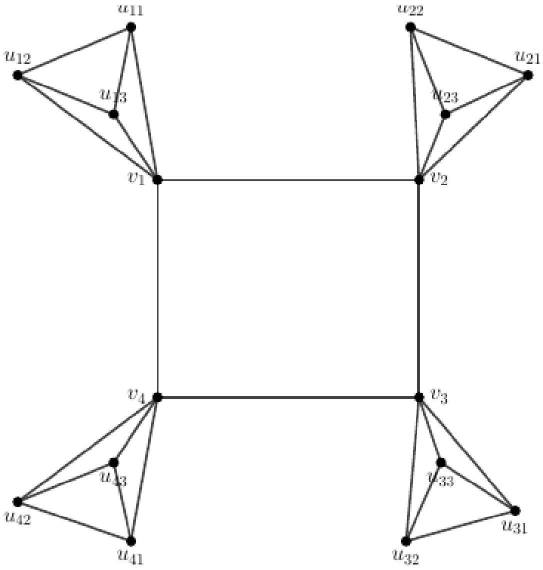

Figure 4. Example of Corona Operation

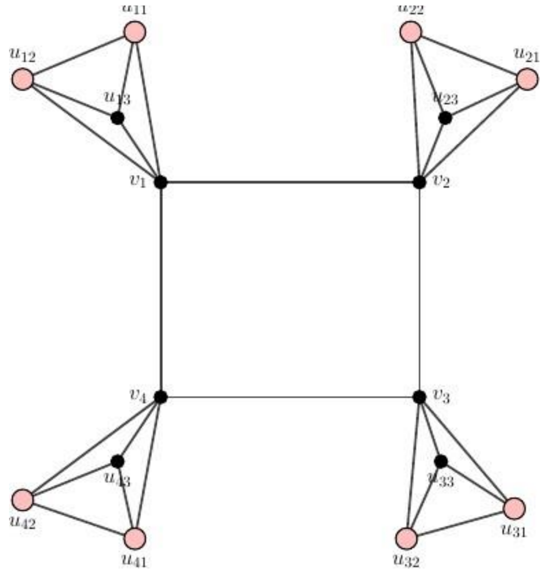

Figure 5. Graph of Corona Operation Results $C_4 \odot C_3$

We have W = $\{u_{11}, u_{12}, u_{21}, u_{22}, u_{31}, u_{32}, u_{41}, u_{42}\}$ then we have representation as follows:

$$r(u_{11}|W) = (0, 1, 3, 3, 4, 4, 3, 3)$$
$$r(u_{12}|W) = (1, 0, 3, 3, 4, 4, 3, 3)$$
$$r(u_{13}|W) = (1, 1, 3, 3, 4, 4, 3, 3)$$
$$r(u_{21}|W) = (3, 3, 0, 1, 3, 3, 4, 4)$$
$$r(u_{22}|W) = (3, 3, 1, 0, 3, 3, 4, 4)$$
$$r(u_{23}|W) = (3, 3, 1, 1, 3, 3, 4, 4)$$
$$r(u_{31}|W) = (4, 4, 3, 3, 0, 1, 3, 3)$$
$$r(u_{32}|W) = (4, 4, 3, 3, 1, 0, 3, 3)$$
$$r(u_{33}|W) = (4, 4, 3, 3, 1, 1, 3, 3)$$
$$r(u_{41}|W) = (3, 3, 4, 4, 3, 3, 0, 1)$$
$$r(u_{42}|W) = (3, 3, 4, 4, 3, 3, 1, 0)$$
$$r(u_{43}|W) = (3, 3, 4, 4, 3, 3, 1, 1)$$
$$r(v_1|W) = (1, 1, 2, 2, 3, 3, 2, 2)$$
$$r(v_2|W) = (2, 2, 1, 1, 2, 2, 3, 3)$$
$$r(v_3|W) = (3, 3, 2, 2, 1, 1, 2, 2)$$
$$r(v_4|W) = (2, 2, 3, 3, 2, 2, 1, 1)$$

Note that the metric dimension of the graph resulting from the corona operation $C_n \odot C_{n-1}$ is $2n$. The following theorem guarantees this.

Theorem 3.2. The metric dimension of the graph resulting from the corona operation $C_n \odot C_{n-1}$ is $2n$.

Proof. We note that G is corona product of $C_n \odot C_{n-1}$, then G consists of $V(G) = \{u_{11}, \cdots, u_{1(n-1)}, u_{21}, \cdots, u_{2(n-1)}, u_{n1}, \cdots, u_{n(n-1)}, v_n\}$. Let $W = \{u_{11}, u_{12}, u_{21}, u_{22}, \cdots u_{n1}, u_{n2}\}$ is resolving set such that $|W| = 2n$. Representation $r(u|W)$ as follows:

$$r(u_{1(n-1)}|W) = (1, \cdots, 1, 3, \cdots, 3, \cdots, 3)$$

$$r(u_{2(n-1)}|W) = (3, \cdots, 3, 1, \cdots, 1, \cdots, 4)$$
$$\vdots$$
$$r(u_{n(n-1)}|W) = (3, \cdots, 3, 4, \cdots, 4, \cdots, 1).$$
$$r(v_1|W) = (1, \cdots, 1, 2, \cdots, 2, \cdots, 2).$$
$$r(v_2|W) = (2, \cdots, 2, 1, \cdots, 1, \cdots, 3).$$
$$\vdots$$
$$r(v_n|W) = (2, \cdots, 2, 3, \cdots, 3, \cdots, 1).$$

Since the coordinates $r(u|W)$ for each vertex are different, $|W| = 2n$ is a resolving set. Next, it will be proven that the resolving set $W = \{u_{11}, u_{12}, u_{21}, u_{22}, \dots, u_{n1}, u_{n2}\}$ is a set with minimum cardinality. Suppose there is a resolving set $W = \{u_{11}, u_{21}, \dots, u_{n1}\}$ such that $|W| = n$. The coordinate $r(u|W)$ of vertex u concerning W is

$$r(u_{12}|W) = (1, 3, \dots, 3).$$
$$r(u_{1(n-1)}|W) = (1, 3, \dots, 3).$$

Since there are $u_{12}, u_{1(n-1)} \in V(G)$ with $u_{12} \neq u_{1(n-1)}$ and $r(u_{12}|W) = r(u_{1(n-1)}|W)$, then $W = \{u_{11}, u_{21}, \dots, u_{n1}\}$ is not a revolving set. Because there is no resolving set with cardinality less than $2n$, W with cardinality $2n$ is the resolving set with minimum cardinality. Therefore, $dim(C_n \odot C_{n-1}) = 2n$.

4 Conclusion

The metric dimension of the graph resulting from the corona operation (C_n $circ$ $C_{\{n-1\}}$) is ($2n$). This result highlights the relationship between the structure of the original graphs and the complexity of their metric dimension when combined through the corona operation. Specifically, it shows that the metric dimension scales linearly with the number of vertices in the initial cycle graph (C_n). Understanding this relationship can be valuable for applications in network design, where the uniqueness of vertex identification is critical.

Suggestions:

1. Further Research: Investigate the metric dimensions of other types of graphs resulting from corona operations with different base graphs. This can help to generalize the findings and explore potential patterns or rules.
2. Algorithm Development: Develop efficient algorithms to compute the metric dimension for large graphs resulting from corona operations. This will be beneficial for practical applications where computational resources are limited.
3. Application in Network Design: Utilize the findings in designing robust and efficient communication networks where unique identification of nodes is necessary. Understanding the metric dimension can improve network navigation and error detection.
4. Educational Tools: Create educational materials and tools to help students and researchers understand and visualize the concept of metric dimensions in corona operations. Interactive graph tools and visual aids can enhance learning and research efficiency.

Reference

[1] Chartrand, G. and L. Lesniak, *Graph and Digraphs*, Wadsworth and Brooks/Cole Advanced Books and Software, Pacific Grove, California, 1986.

[2] Chartrand, G. and O. R. Oellermann, Applied and Algorithmic Graph Theory, International Series in Pure and Applied Mathematics, McGraw-Hill Inc, California, 1993

[3] Gallian, J. A. Dynamic Survey of Graph Labeling, *The Electronic Journal of Combinatorics*, 16, (2014), #DS6, pp.1-384.

[4] Slater, P.J., Leaves of trees, *Congr. Numer.*, 14 (1975), 549 -559

[5] Slater, P.J., Dominating and reference sets in graphs, *J. Math. Phys. Sci.*, 22(1988), 445-455.

[6] Harary, F., and Melter, R.A., On the metric dimension of a graph, *Ars. Combin.*,2 (1976), 191-195.

[7] Chartrand, G., L. Eroh, M.A. Johnson, and O.R. Oellermann, Resolvability in graphs and the metric dimension of a graph, *Discrete Appl. Math.*, 105 (2000), 99 - 113.

[8] Buczkowski, P. S., Chartrand, G., Poisson, C., and Zhang, P., On k-dimensional Graphs and their bases, Periodica Math. Hung. 46(1), (2003), 9-15.

[9] Bača, M., E.T. Baskoro, A.N.M. Salman, S.W. Saputro, D. Suprijanto, The metric dimension of regular bipartite graphs, *Bull. Math. Soc. Sci. Math*. Roumanie Tome, 54(102) No. 1, (2011), 15–28.

[10] Khuller S., Ragavachari B., Rosenfeld A., Landmarks in Graphs, *Discrete Applied Mathematics*, 70, (1996), 217-229.

[11] Bharati R., Indra R., J. A. Cynthia. and Paul M., *On minimum metric dimension*, Proceedings of the Indonesia-Japan Conference on Combinatorial Geometry and Graph Theory, Bandung, Indonesia, 2003 pp. 13-16.

[12] Ahmad, S., M. A. Chaudhry, I. Javaid and M. Salman, On the metric dimension of generalized Petersen graphs, *Quaestiones Mathematicae*, 36, Issue 3, (2013), 421 – 435.

[13] Shahida.A.T, and M.S.Sunitha, On the Metric Dimension of Joins of Two Graphs, *International Journal of Scientific and Engineering Research*, 5, Issue 9, (2014), 33 – 38.

[14] Bailey, R. F. and K. Meagher, On the metric dimension of Grassmann graphs, *Discrete Mathematics and Theoretical Computer Science*, 13:4, (2011), 97–104.

[15] Fitriani, F., & Cahyaningtyas, S. (2021). Graf Dual Antiprisma Dan Dimensi Metriknya. E-Jurnal Matematika, 313.

[16] Janan, T., & Janan, S. (2022). Dimensi Metrik dari Graf Jaring Laba-Laba. Proximal: Jurnal Penelitian Matematika dan Pendidikan Matematika, 181-190.

[17] Silalahi, R., & Mulyono. (2023). Metric dimensions and partition dimensions of a multiple fan graph. Formosa Journal of Science and Technology, 81-88.

[18] Azka, D., Palupi, D., & Sutjijana, A. (2022). Dimensi Metrik Lokal dari Hasil Perkalian Kuat Graf Bintang. Journal Fourier, 49-58.

[19] Lathifah, F. (2024). Dimensi Metrik Lokal pada Graf Garis dari Graf Persahabatan. Jurnal Kajian dan Terapan Matematika, 53-58

[20] Kusmayadi, T. A., Kuntari, S., Rahmadi, D., & Lathifah, F. A. (2016). On the strong metric dimension of some related wheel graph. Far East Journal of Mathematical Sciences , 1325-1334.

[21] Rahmadi, D., & Susanti, Y. (2022). The k-metric dimension of double fan graph. Quadratic: Journal of Innovation and Technology in Mathematics and Mathematics Education, 31-35.

[22] Rahmadi, D. (2024). Dimensi metrik campuran pada graf double fan. Jurnal Diferensial, 52-56.

[23] Nie, K., & Xu, K. (2024). The doubly metric dimensions of cactus graphs and block graphs. J Comb Optim.