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The Comparison of Parameter Estimation in Exponential Distribution using the Maximum Likelihood Method and Bayesian Method

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Abstract. This study aims to compare the parameter estimation process on the exponential distribution using the maximum likelihood method and the Bayesian method on service data at Bank BRI Atambua Branch. The result of processing data on 100 service time data obtained parameter values for the maximum likelihood method $\hat{\theta}$= 0,050942, and the Bayesian $\hat{\theta}$= 1,0008. By using the Akaike Information Criterion (AIC) feasibility test for these parameters, the AIC value for the maximum likelihood method was 797,4122, and the AIC value for the Bayesian method was 3.931,1002 so it can be concluded that the maximum likelihood method is better used to estimate parameters than Bayesian method.

Keywords: Exponential Distribution, Maximum Likelihood Method, Bayesian Method, AIC.

1 Introduction

Parameter estimation estimates population characteristics using sample characteristics [1]. The population is usually enormous, so to know its characteristics through estimation by conducting a survey of samples taken randomly from the population. The results of the sample characteristics from the survey are used to estimate the characteristics of the population. The sample used in the survey is a sample that truly represents the population.

The exponential distribution is one of the distributions that have much practical value, especially in matters related to time, for example, waiting time, the lifetime of a device or the length of time until a device stops functioning, the length of telephone conversations, and so on [1]. With this quantitative research type, the researcher chose the service time modeling at the Atambua Branch of BRI as a case example. The data taken in this study were arrival time, service start time, and service finish time. The exponential distribution is continuous and is a particular form of the gamma distribution with two parameters, namely α and β.

Parameter estimation can be done using two methods, namely the Maximum Likelihood Estimation (MLE) method and the Bayesian method. The MLE method is a method that maximizes the likelihood function for a population with a known distribution [2]. MLE can evaluate the quantitative variance and correlation of spectral response patterns when classifying unknown pixels [1]. Bayes' theorem was discovered in 1763, perfecting the conditional probability theorem, which is limited to only two events to extend to n events. The name Bayes' theorem is taken from the name of the inventor of the theorem, namely Reverend Thomas Bayes

(1702-1761), an English Presbyterian minister. The Bayesian method is another method that researchers often use in estimating the parameters of a distribution [3].

In the previous study by [2], applying examples of earthquake data around the world, the results of research on 20 earthquake data produced a parameter estimate value for the data given, namely the maximum likelihood method, $\hat{\theta}$ = 0,00207619 and the Bayesian method $\hat{\theta}$ = 0,00220057. A comparison of the two methods using the AIC (Akaike Information Criterion) adequacy test concluded that the maximum likelihood method is the best method for estimating exponential distribution parameters. In line with this research, the researchers chose service time data samples at BRI's Atambua Branch for comparison of parameter estimates on the exponential distribution using the maximum likelihood method and the Bayesian method with the AIC feasibility test to obtain the best method for estimating the parameters of the exponential distribution.

This study aims to compare the estimated values of parameters from exponentially distributed data using the maximum likelihood method and the Bayesian method. Determine the best estimation method based on AIC criteria on service time data at BRI Atambua Branch.

2 Theoretical Basis

2.1 Parameter Estimation

Parameter estimation estimates population characteristics (parameters) using sample characteristics (statistics). The population is usually huge, so it is difficult to know its characteristics through a census. Census is very uneconomical in terms of time, effort, and cost. Therefore, we can make predictions by surveying samples taken randomly from the population. Then we use the results of the survey's sample characteristics to estimate the population's characteristics. The sample used in the survey is a sample that truly represents the population.

In statistics, the statistical value $\hat{\theta}$. It is used to estimate the parameter θ so that $\bar{x}$ is used to estimate μ, s^2 is used to estimate σ and $\hat{p}$ It is used to estimate p. Because its nature is estimation or approximation, the value of the $\hat{\theta}$ Estimator will, of course, not be the same as the parameter θ value.

2.2 Exponential Distribution

A probability distribution is said to be exponentially distributed with one parameter $X \sim Exp(\theta)$, if the distribution has a probability density function

$$f(x) = \theta e^{-x\theta} \quad , x > 0 \tag{1}$$

with mean $E(x) = \frac{1}{\theta}$ and variance $Var(x) = \frac{1}{\theta^2}$.

2.3 Gamma Distribution

The Gamma distribution is one of the exponential distribution families. A random variable is said to be Gamma distributed with parameters β and α if and only if the density function has the form.

$$f(\theta) = \frac{\beta^{\alpha}\theta^{\alpha-1}e^{-\beta\theta}}{\Gamma(\alpha)} \tag{2}$$

with mean $E(\theta) = \frac{\alpha}{\beta}$ and variance $Var(x) = \frac{1}{\alpha\beta^2}$.

2.4 Maximum Likelihood Estimation (MLE)

[4] said that, suppose $x_1, x_2, \cdots, x_n$ is a random variable from the population whose probability density function is denoted by $f(x, \theta)$, where θ is the unknown parameter. Then the likelihood function of the sample is

$$\begin{aligned} L(x_1, x_2, \cdots, x_n; \theta) &= f(x_1, \theta) \ldots f(x_n, \theta) \cdots f(x_n; \theta) \\ &= \prod_{i=1}^{n} f(x_i \theta) \\ &= L(\theta \,|\, x_1, x_2, \cdots, x_n) \\ &= L(\theta \,|\, X) \end{aligned} \tag{3}$$

Next, there is $\hat{\theta}(\bar{x})$ such that $L(\theta \,|\, X)$ reaches a maximum and is called the maximum likelihood estimator. This means $\hat{\theta}$ Is the value of θ that satisfies

$$L(x_1, x_2, \cdots, x_n |\, \hat{\theta}) = \max_{\theta \epsilon \Omega} L(x_1, x_2, \cdots, x_n;\ \theta) \tag{4}$$

2.5 Bayesian Method

Bayesian is a method that views parameters as variables that describe initial information about parameters before observations are made and expressed in a prior distribution. Meanwhile, the parameters of the prior distribution in this study have been determined using the Gamma distribution. Furthermore, posterior information (posterior distribution) is obtained, a combination of two sources of information regarding the parameters of the statistical model, namely the likelihood of the sample distribution and the initial information (prior distribution). The results are expressed as a posterior distribution, which becomes the basis for the Bayesian method [4].

Prior Distribution

In the Bayesian method, selecting the prior distribution indicates uncertainty about the unknown parameters. In Bayesian inference, the parameter θ is treated as a variable, and it will have a value in a domain with density $f(\theta)$, and this density will be named as the prior distribution of θ, with this preliminary information, it will be combined with the sample data used in form posteriorly. If a population follows a particular distribution with a parameter in it, for example, the parameter θ, then the parameter θ itself also follows a particular probability distribution called the prior distribution. The prior distribution is denoted by $\pi(\theta)$, so it can be written $\theta \sim \pi(\theta)$ [4].

Posterior Distribution

[4] the posterior distribution is the conditional density function θ if the observed value x is known. In the Bayesian method, the inference is based on the posterior distribution. So that the posterior distribution is expressed as follows:

$$\pi(\theta|\underline{x}) \propto L(\theta|\underline{x})\pi(\theta) \quad (5)$$

where

$\pi(\theta|\underline{x})$: posterior function

$L(\theta|\underline{x})$: maximum likelihood function

$\pi(\theta)$: prior distribution

2.6 Due Diligence with Akaike Information Criterion (AIC)

Statistical modeling that compares several models is usually followed by a model goodness-of-fit test. This is done to ensure that one of the models is the best. Several goodness-of-fit tests, such as the Kolmogorov Smirnov test, MSE (Mean Square Error), and AIC (Akaike Information Criterion) values, have been used for this comparison. In this study, the AIC value will be used to obtain the best method for estimating the parameters of the exponential distribution. The AIC value depends on the log-likelihood value of a probability density function, and the well-known AIC value can be used as a guide to determine the best method for estimating parameters [5]. The formula can determine the AIC value:

$$\text{AIC} = -2l + 2p \quad (6)$$

Where l = log-likelihood and p = a number of parameters.

3 Method

This research took a case at Bank Rakyat Indonesia (BRI) Atambua Branch for five days in December 2022. This research is a quantitative study. The data taken is in the form of customer arrival time data, service start time data, and service completion time data. This research was conducted using literature studies and observations. The data analysis technique used the Maximum Likelihood Estimation (MLE), Bayesian methods, and R software. The stages in the calculation process for the MLE method and the Bayesian method are as follows:

3.1 Determining Parameter Estimation Using The Maximum Likelihood Method

In line with research [6], the estimation of exponential distribution parameters using the Maximum Likelihood Method can be done with the following steps:

a. Create an exponential distribution likelihood function.
b. Make the function in the form *ln*.
c. Make partial derivatives of the parameter θ and equate to zero.
d. From the partial derivatives concerning θ, the estimator θ can be obtained.

3.2 Determining Parameter Estimation Using The Bayesian Method

In line with research [4], estimating the parameters of the exponential distribution using the Bayesian Method can be carried out with the following steps:

a. Create an exponential distribution likelihood function.
b. Create a primary distribution function.
c. Make the distribution of the posterior function.
d. Simulation of the posterior distribution function with several iterations using the software.
e. The parameter is determined with the smallest value in the iteration it has.

3.3 Determine Due Diligence Using AIC

Based on [2], parameter comparison using the maximum likelihood method and the Bayesian method can be carried out with the following steps:

a. We obtained log-likelihood values for each parameter from the maximum likelihood and Bayesian methods.
b. Get the AIC value.
c. Comparing the smallest AIC value is the best parameter value.

4 Result and Discussion

4.1 Estimation of Exponential Distribution Parameters Using The Maximum Likelihood Method

Create the Exponential Distribution Likelihood Function.

Before determining the maximum likelihood estimate of the exponential distribution, what must be determined first is to determine the likelihood function of the exponential distribution.

Suppose $f(x_1, x_2, \cdots x_n; \theta)$ at a point $x_1, x_2, \cdots x_n$ then the probability density function is

$$f(x_1, x_2, \cdots x_n|\theta) = f(x_1;\theta)\, f(x_2;\theta), \cdots f(x_n;\theta) \text{ since } f(x,\theta) = \theta e^{-x\theta}$$

thus

$$\begin{aligned} L(\theta|\underline{x}) &= f(x_1 \mid \theta) f(x_2 \mid \theta) \cdots f(x_n \mid \theta) \\ &= \prod_{i=1}^{n} f(x_i \mid \theta) \\ &= \left(\theta e^{-x_1\theta}\right)\left(\theta^{-x_2\theta}\right)\cdots\left(\theta^{-x_n\theta}\right) \\ &= \theta^n e^{-\theta \sum_{i=1}^{n} x_i} \end{aligned}$$

Since $L(\theta|\underline{x}) = f(x_1, x_{2,\ldots} x_n; \theta)$, then the likelihood function of the exponential distribution is

$$L\left(\theta|\underline{x}\right) = \theta^n e^{-\theta \sum_{i=1}^{n} x_i}. \quad (7)$$

We are creating the Likelihood Function of the Exponential Distribution in *ln* Form.

The maximum likelihood estimate θ is a value. $\hat{\theta}$ That maximizes the likelihood function $L(\theta|\underline{x})$ therefore $(\hat{\theta}|\underline{x}) \geq L(\theta|\underline{x})$. For $\hat{\theta}$ which maximize $L\left(\theta|\underline{x}\right)$ then $\hat{\theta}$ It will also be maximum. In order to obtain the maximum likelihood estimate from the likelihood function, a Newton-Raphson likelihood $ln(\theta|\underline{x})$ must be formed with the notation $L(\theta|\underline{x})$. The Newton-Raphson likelihood function is $ln\left(L(\theta|\underline{x})\right) = ln(\theta^n e^{-\sum_{i=1}^{n} x_i \theta})$. By using Newton-Raphson $L(\theta|\underline{x})$ The likelihood estimator is directly obtained from $\frac{dL(\theta|\underline{x})}{d\theta} = 0$ because

$$\begin{aligned} ln\left(L(\theta|\underline{x})\right) &= ln\left(\theta^n e^{-\theta \sum_{i=1}^{n} x_i}\right) \\ &= ln\theta^n + lne^{-\theta \sum_{i=1}^{n} x_i} \\ &= n\, ln\, \theta - \theta \textstyle\sum_{i=1}^{n} x_i. \end{aligned} \quad (8)$$

Make a partial derivative concerning θ and equate it to zero.

$$\begin{aligned} \frac{dl(\theta|\underline{x})}{d\theta} &= \frac{d}{d\theta}\left[n\, ln\, \theta - \theta \sum_{i=1}^{n} x_i\right] \\ &= \frac{d}{d\theta}(n\, ln\, \theta) - \frac{d}{d\theta}\left(\theta \sum_{i=1}^{n} x_i\right) \\ &= \frac{n}{\theta} - \sum_{i=1}^{n} x_i \end{aligned} \quad (9)$$

$$\begin{aligned} \frac{dL(\theta|\underline{x})}{d\theta} &= 0 \Leftrightarrow \frac{n}{\theta} - \sum_{i=1}^{n} x_i = 0 \\ \frac{n}{\theta} &= \sum_{i=1}^{n} x_i \\ \theta &= \frac{\boldsymbol{n}}{\sum_{i=1}^{n} \boldsymbol{x_i}} \end{aligned} \quad (10)$$

Thus the estimator θ is obtained. Therefore the maximum likelihood estimate for $\hat{\theta} = \frac{n}{\sum_{i=1}^{n} x_i}$.

4.2 Estimation of Exponential Distribution Parameters using The Bayesian Method

Create The Prior Distribution Function

The likelihood function of the same exponential distribution determines the prior distribution. The prior distribution is denoted by $\pi(\theta)$, so it is written $\theta \backsim \pi(\theta)$. The prior distribution has been determined using the Gamma function with two parameters: α and β. The opportunity density function can be expressed in the following form of equation (11).

$$\pi(\theta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\theta\beta}, \theta > 0 \qquad (11)$$

Create The Posterior Distribution Function

When the likelihood function is multiplied by the prior distribution, we get

$$\begin{aligned}\pi(\theta|\underline{x}) = \pi(\theta)L(\theta|\underline{x}) &= \frac{\beta^{\alpha}\theta^{\alpha-1}e^{-\beta\theta}}{\Gamma(\alpha)}\left(\theta^{n}e^{-\theta\sum_{i=1}^{n}x_i\theta}\right)\\ &= \frac{\beta^{\alpha}}{\Gamma(\alpha)}\theta^{n+\alpha-1}e^{(-\theta(\sum_{i=1}^{n}x_i+\beta))}\\ &= \frac{\beta^{\alpha}}{\Gamma(\alpha)}\theta^{n+\alpha-1}e^{-(\sum_{i=1}^{n}x_i\theta+\beta\theta)}\end{aligned} \qquad (12)$$

then the Bayesian estimation is given in the following equation:

$$\pi(\theta|\underline{x}) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\theta^{n+\alpha-1}e^{-\theta(\sum_{i=1}^{n}x_i+\beta)}.$$

Alternatively, it can be expressed by the following equation 4.8:

$$(\theta|\underline{x}) \sim gamma\ (n+\alpha, \sum_{i=1}^{n}x_i+\beta) \qquad (13)$$

Application of Sample Data for Parameter Estimation using The Maximum Likelihood Method and The Bayesian Method

The data used in this research is service time data at the Atambua Branch of BRI.

Table 1. Service time data.

Day	Customer (Person)	Service Time (Minute)
Monday	20	374
Tuesday	20	421
Wednesday	20	425
Thursday	20	378
Friday	20	365
Amount	100	1963

This data was taken from Monday, 12 December 2022 to Friday, 16 December 2022 at 08.00-16.00. This data will be used to apply the maximum likelihood and Bayesian methods to estimate the parameters of the one-parameter exponential distribution.

Parameter Estimation Using The Maximum Likelihood Method.

We know that

$n = 100$

$\sum_{i=1}^{n} x_i = 1963$ Menit

parameter estimation using the maximum likelihood method, we obtained

$\hat{\theta} = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{100}{1963} = 0{,}0509.$

Thus parameter estimation is obtained. $\hat{\theta} = 0{,}0509$.

The value $\hat{\theta} = 0{,}0509$, then substituted into the probability density function of the exponential distribution in equation (11)

$f(x) = \theta e^{-x\theta} = 0{,}05094243505\ e^{-0{,}05094243505x}$.

Parameter Estimation Using The Bayesian Method.

By substituting n and $\sum_{i=1}^{n} x_i$ Into the maximum likelihood function, we get:

$L\left(\theta \middle| \underline{x}\right) = \theta^n e^{-\sum_{i=1}^{n} x_i \theta}$

$L\left(\theta \middle| \underline{x}\right) = \theta^{100} e^{-(1963\theta)}$.

The prior distribution has been determined using the Gamma function in Equation 11. Next, by using $\left(\alpha = \frac{n}{2}\right)$ and $\left(\beta = (n-1)\frac{\sum_{i=1}^{n} x_i^2}{n}\right)$ we get $\alpha = 50$ and $\beta = 15{,}012813$. These α and β values are then substituted into the prior distribution and obtained.

$$\pi(\theta) = \frac{\beta^\alpha \theta^{\alpha-1} e^{-\theta\beta}}{\Gamma(\alpha)}$$

$$\pi(\theta) = \frac{15{,}012813^{50} \theta^{50-1} e^{-15{,}012813\theta}}{\Gamma(50)}$$

$$\pi(\theta) = \frac{15{,}012813^{50} \theta^{49} e^{-15{,}012813\theta}}{\Gamma(50)}.$$

The density function for the posterior distribution of service time data at the Atambua Branch of BRI is

$$\pi\left(\theta \middle| \underline{x}\right) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{n+\alpha-1} e^{-\theta\left(\sum_{i=1}^{n} x_i + \beta\right)}$$

$$\pi\left(\theta \middle| \underline{x}\right) = \frac{15{,}012813^{50}}{\Gamma(50)} \theta^{100+50-1} e^{-\theta(1963+15{,}012813)}$$

$$\pi\left(\theta \middle| \underline{x}\right) = \frac{15{,}012813^{50}}{\Gamma(50)} \theta^{149} e^{-(15{,}0147190\theta)}$$

Therefore we get the posterior distribution $\left(\theta \middle| \underline{x}\right) \sim \Gamma(150, 15{,}0147)$.

Using R programming, the posterior distribution function is simulated for 3000 iterations so that the parameter value $\theta = 1{,}000832$ is obtained.

The parameter values generated by the Bayesian method can be substituted into the probability density function of the exponential distribution $f(x) = \theta e^{-x\theta} = 1{,}000832 e^{-(1{,}000832x)}$.

4.3 Feasibility Test Using Akaike Information Criterion (AIC).

Based on the calculation of the parameter values, the exponential distribution parameter values are obtained using the MLE method and the Bayesian method, as presented in Table 2.

Table 2. Parameter values using MLE and Bayesian methods.

Method	Parameter Value
MLE	0,0509
Bayesian	1,008

From the parameter values in Table 3.2 above, it can be solved using the AIC formula.
AIC value for the MLE method is calculated using equation (6) as follows:

$AIC = -2l + 2p$

$AIC = -2(n \times ln(\theta) - \theta \sum_{i=1}^{n} x_i) + 2p$

$= -2(100 \times ln(0{,}050942) - 0{,}050942 \times 1963) + 2(1)$

$= -2(100 \times (-2.977067) - 99.999146) + 2$

$= -2(-297.706 - 99.999146) + 2$

$= -2(-397.706146) + 2$

$= 797{,}4122.$

Furthermore, the AIC value for the Bayesian method is calculated by equation (6) as follows:

$AIC = -2(n \times ln(\theta) - \theta \sum_{i=1}^{n} x_i) + 2p$

$= -2(100 \times ln(1{,}000832) - 1{,}000832 \times 1963) + 2(1)$

$= -2\big((100 \times 0{,}000831) - 1.964633216\big) + 2$

$= -2(0{,}0831 - 1.964633216) + 2$

$= -2(-1.964550116) + 2$

$= 3.931{,}1002.$

Table 3 AIC Values for the Exponential Distribution of the MLE and Bayesian Methods

Table 3. AIC value for the exponential distribution and Bayesian methods.

Method	AIC
MLE	797,4122
Bayesian	3.931,1002

From Table 3, it can be seen that the MLE method has a smaller AIC value than the Bayesian method. Therefore, it can be concluded that the MLE method is an excellent method to use for estimating exponential distribution parameters with the probability density function of the exponential distribution of service data at BRI Atambua Branch having an average value of $E(X) = \frac{1}{\theta}$ = 19,6301 and variance $Var(X) = 385{,}3434$..

5 Conclusion

Based on data processing, parameter estimation using the maximum likelihood method obtained an estimated parameter value of $\hat{\theta} = 0{,}05094$ while using the Bayesian method obtained an estimated parameter value of

$\hat{\theta} = 1{,}0008$.. Both methods were tested using the AIC feasibility test showing that the maximum likelihood method is the best for estimating exponential distribution parameters.

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